In any triangle, the feet of the altitudes, the midpoints of the sides, and the midpoints of the line segments connecting the vertices with the orthocenter all lie on a circle.

In order to prove the existence of such a circle, we break the proof into three steps.
Lemma 1:

In triangle $ABC$, points $A_H$ and $B_H$ are on $BC$ and $AC$, respectively, so that $AA_H$ is perpendicular to $BC$ and $BB_H$ is perpendicular to $AC$. Prove that triangle $A_HB_HC$ is similar to triangle $ABC$.

First notice that the triangles CAB and $CB_HA_H$ share angle C. Then, it is clear that angles $AB_HB$ and $AA_HB$ are equal, as they are both right angles. Thus quadrilateral $AB_HA_HB$ is cyclic, and therefore angle $ABC$ supplements angle $AB_HA_H$. Note that angle $AB_HA_H$ supplements $CB_HA_H$ as well.
Thus, angles $CB_{AH}$ and $ABC$ are congruent, and therefore triangle $A_{BH}B_{CH}C$ is similar to triangle $ABC$.

**Lemma 2:**

In triangle $ABC$, points $A_{BH}$, $B_{CH}$ and $C_{AH}$ lie on lines $BC$, $AC$, and $AB$ respectively, so that $AA_{BH}$ is perpendicular to $BC$, $BB_{CH}$ is perpendicular to $AC$, and $CC_{AH}$ is perpendicular to $AB$. Additionally, point $M$ lies on $AB$ such that segment $AM$ is congruent to $BM$. Prove that quadrilateral $A_{BH}B_{CH}C_{AH}M$ is cyclic.

By applying the previous proof, we find that angles $CB_{AH}$, $C_{AH}$, and $AB_{CH}$ are congruent. We will call this angle value $\beta$.

It is obvious that angles $CB_{AH}$, $AB_{CH}$, and $C_{AH}B_{AH}$ are supplementary and that angle $C_{AH}B_{AH}$ is equal to $180^\circ-2\beta$.

Additionally, in triangle $AA_{BH}B$, $M$ is the midpoint of the hypotenuse $AB$. Thus follows that lines $A_{BH}M$ and $MB$ are congruent.

Since lines $A_{BH}M$ and $MB$ are congruent, angle $MBA_{BH}$ and $MA_{BH}B$ are congruent, equal to angle value $\beta$.

Using that fact, $BAMA_{BH}$ is equivalent to $180^\circ-2\beta$, and, from this, angle $BAMA_{BH}$ is equal to $2\beta$. 
From this, we can conclude that the quadrilateral $MCAH$ is cyclic, and therefore, the feet of the altitudes perpendiculars and the midpoints lie on the same circle.

**Lemma 3:**

In triangle $ABC$, points $H_a$, $H_b$, and $H_c$ lie on lines $BC$, $AC$, and $AB$ respectively such that $AH_a$ is perpendicular to $BC$, $BH_b$ is perpendicular to $AC$, and $CH_c$ is perpendicular to $AB$. Additionally, point $H$ lies on the intersection of lines $AH_a$, $BH_b$, and $CH_c$. Finally, point $O_c$ lies on line $HC$ such that line $HO_c$ is congruent to line $O_cC$. Prove that points $H_c$, $H_b$, $H_a$, and $O_c$ are concyclic.

\[
\angle H_bH_cH_a = 180° - 2C \implies O_c \text{ lies on the same circle as the midpoints and feet of the altitudes of triangle ABC because } H_aH_cH_b \text{ is cyclic.}
\]

This argument can be repeated for $O_a$ and $O_b$.

Also, $O_a$, $O_b$, and $O_c$ are sometimes referred to as the midpoints of the segments from the orthocenter to the vertices, and they are:

\[
\angle H_bHH_a = 360° - C - 90° - 90° = 180° - C \implies H_bHH_aC \text{ is cyclic.}
\]

$O_c$ is the circumcenter of $H_aH_bC$, and thus the center of the circle circumscribed around $H_bHH_aC$. So $CO_c$ and $O_cH$ are both radii and thus equal in length, and $O_c$ is the midpoint of $HC$. Identical proofs for $O_a$ and $O_b$ follow.
Therefore, all nine points lie on the same circle.